

Nonlinear control systems

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4.41

Consider the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = 2x_1x_2 + 3t + 2 - 3x_1 - 2(t+1)x_2.$$

(a)

We are given $x_1(t) = t$ and $x_2(t) = 1$. Check both equations:

For the first equation:

$$\dot{x}_1 = 1, \quad x_2 = 1 \quad \Rightarrow \quad \dot{x}_1 = x_2,$$

so it holds.

For the second equation:

$$\dot{x}_2 = 0 \quad (\text{since } x_2 = 1 \text{ is constant}).$$

Now compute the right-hand side with $x_1 = t$, $x_2 = 1$:

$$\begin{aligned} 2x_1x_2 + 3t + 2 - 3x_1 - 2(t+1)x_2 &= 2(t)(1) + 3t + 2 - 3t - 2(t+1)(1) \\ &= 2t + 3t + 2 - 3t - 2t - 2 = (2t + 3t - 3t - 2t) + (2 - 2) = 0. \end{aligned}$$

Thus $\dot{x}_2 = 0$ matches the right-hand side.

Therefore $\boxed{x_1(t) = t, x_2(t) = 1}$ is indeed a solution.

(b)

We want to show that if $x(0)$ is sufficiently close to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $x(t)$ approaches $\begin{bmatrix} t \\ 1 \end{bmatrix}$ as $t \rightarrow \infty$.

Define the error variables:

$$y_1(t) = x_1(t) - t, \quad y_2(t) = x_2(t) - 1.$$

We aim to show that $(y_1, y_2) \rightarrow (0, 0)$ as $t \rightarrow \infty$ for small initial conditions.

Differentiate:

$$\dot{y}_1 = \dot{x}_1 - 1 = x_2 - 1 = y_2.$$

For y_2 :

$$\dot{y}_2 = \dot{x}_2 = 2x_1x_2 + 3t + 2 - 3x_1 - 2(t+1)x_2.$$

Substitute $x_1 = y_1 + t$, $x_2 = y_2 + 1$:

$$\begin{aligned}\dot{y}_2 &= 2(y_1 + t)(y_2 + 1) + 3t + 2 - 3(y_1 + t) - 2(t+1)(y_2 + 1) \\ &= 2(y_1y_2 + y_1 + ty_2 + t) + 3t + 2 - 3y_1 - 3t - 2(t+1)y_2 - 2(t+1) \\ &= 2y_1y_2 + 2y_1 + 2ty_2 + 2t + 3t + 2 - 3y_1 - 3t - 2ty_2 - 2y_2 - 2t - 2.\end{aligned}$$

Cancel terms: - Constant terms: $2t + 3t - 3t - 2t - 2 + 2 = 0$. - Linear y_1 : $2y_1 - 3y_1 = -y_1$.
- Linear y_2 : $2ty_2 - 2ty_2 - 2y_2 = -2y_2$. - Nonlinear: $2y_1y_2$.

Thus the error dynamics are

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = -y_1 - 2y_2 + 2y_1y_2. \end{cases}$$

The linearisation about the origin $(y_1, y_2) = (0, 0)$ is

$$\dot{y} = Ay, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\lambda I - A) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2,$$

so both eigenvalues are $\lambda = -1 < 0$. Hence the linearised system is asymptotically stable.

By the Lyapunov linearisation theorem, the origin of the nonlinear error system is locally asymptotically stable. Therefore, if $y(0)$ is sufficiently small (i.e., $x(0)$ close to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$), the solution $(y_1(t), y_2(t))$ will converge to $(0, 0)$ as $t \rightarrow \infty$.

Consequently,

$$y_1(t) \rightarrow 0 \Rightarrow x_1(t) - t \rightarrow 0 \Rightarrow x_1(t) \rightarrow t,$$

and

$$y_2(t) \rightarrow 0 \Rightarrow x_2(t) - 1 \rightarrow 0 \Rightarrow x_2(t) \rightarrow 1.$$

$$\text{Thus } x(t) \text{ approaches } \begin{bmatrix} t \\ 1 \end{bmatrix} \text{ as } t \rightarrow \infty \text{ for initial conditions sufficiently close to } \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

8.6

We apply the center manifold theorem to each system. In each case, we linearize about the origin, identify the center and stable subspaces, compute the center manifold (up to required order), and reduce the dynamics to the center manifold to determine stability.

$$(1) \quad \dot{x}_1 = -x_2^2, \quad \dot{x}_2 = -x_2 + x_1^2 + x_1x_2$$

Linearization: $J = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$, eigenvalues 0 (center $u = x_1$) and -1 (stable $v = x_2$). Center manifold: $v = h(u) = u^2 + u^3 + O(u^4)$. Reduced dynamics: $\dot{u} = -h(u)^2 = -u^4 + O(u^5)$. Since $\dot{u} < 0$ for $u \neq 0$, the origin is asymptotically stable.

$$\text{Asymptotically stable.}$$

(2) $\dot{x}_1 = ax_1^2 - x_2^2, \quad \dot{x}_2 = -x_2 + x_1^2 + x_1x_2, \quad a \neq 0$

Linearization same as (1): center $u = x_1$, stable $v = x_2$. Center manifold: $v = h(u) = u^2 + (1 - 2a)u^3 + O(u^4)$. Reduced dynamics: $\dot{u} = au^2 - u^4 + \dots$. For small u , $\dot{u} \approx au^2$.
- If $a < 0$, $\dot{u} < 0$ for all $u \neq 0 \rightarrow$ asymptotically stable. - If $a > 0$, $\dot{u} > 0$ for $u \neq 0 \rightarrow$ unstable.

Stable if $a < 0$, unstable if $a > 0$.

(3) $\dot{x}_1 = -x_2 + x_1x_3, \quad \dot{x}_2 = x_1 + x_2x_3, \quad \dot{x}_3 = -x_3 - (x_1^2 + x_2^2) + x_3^2$

Linearization: eigenvalues $\pm i$ (center) and -1 (stable). Center variables: x_1, x_2 ; stable: x_3 . Center manifold: $x_3 = h(x_1, x_2) = -x_1^2 - x_2^2 + O(3)$. Reduced dynamics:

$$\dot{x}_1 = -x_2 - x_1^3 - x_1x_2^2 + \dots, \quad \dot{x}_2 = x_1 - x_1^2x_2 - x_2^3 + \dots$$

Lyapunov function $V = x_1^2 + x_2^2$ gives $\dot{V} = -2(x_1^2 + x_2^2)^2 \leq 0$. Hence origin is asymptotically stable.

Asymptotically stable.

(4) $\dot{x}_1 = x_1^2x_2, \quad \dot{x}_2 = -x_1^3 - x_2$

Linearization: eigenvalues 0 and -1 . Center $u = x_1$, stable $v = x_2$. Center manifold: $v = h(u) = -u^3 + O(u^4)$. Reduced dynamics: $\dot{u} = u^2h(u) = -u^5 + O(u^6)$. Since $\dot{u}$ has opposite sign to u (i.e., $u\dot{u} = -u^6 < 0$), origin is asymptotically stable.

Asymptotically stable.

(5) $\dot{x}_1 = x_1x_3^2, \quad \dot{x}_2 = -x_2 - x_1^2 + 2x_1^3$

Correction: the third variable is likely a typo; we treat the system as $\dot{x}_1 = x_1x_2^2, \dot{x}_2 = -x_2 - x_1^2 + 2x_1^3$. Linearization: eigenvalues 0 and -1 . Center $u = x_1$, stable $v = x_2$. Center manifold: $v = h(u) = -u^2 + 2u^3 + O(u^4)$. Reduced dynamics: $\dot{u} = uh(u)^2 = u^5 + O(u^6)$. Since $\dot{u}$ has the same sign as u , the origin is unstable.

Unstable.

(6) $\dot{x}_1 = -x_1 + x_3^2(x_1 + x_3 - 1), \quad \dot{x}_2 = x_3^2(x_1 + x_3 - 1)$

Correction: treat x_3 as x_2 . The system becomes $\dot{x}_1 = -x_1 + x_2^2(x_1 + x_2 - 1), \quad \dot{x}_2 = x_2^2(x_1 + x_2 - 1)$. Linearization: eigenvalues -1 and 0 . Center $u = x_2$, stable $v = x_1$. Center manifold: $v = h(u) = -u^2 - u^3 + O(u^4)$. Reduced dynamics: $\dot{u} = u^2(h(u) + u - 1) = -u^2 + u^3 - u^4 + \dots$. For $u > 0$, $\dot{u} < 0$ (approaches 0); for $u < 0$, $\dot{u} < 0$ (moves away). Hence origin is unstable (saddle-node).

Unstable.

(7) $\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 + \frac{ax_3^2}{1+x_1^2}, \quad a \neq 0$

Correction: treat x_3 as x_1 . The system becomes $\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 + \frac{ax_1^2}{1+x_1^2}$. Linearization: $J = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$, eigenvalues 0 and -1 . Center $u = x_1$, stable $v = x_2$. Center manifold: $v = h(u) = au^2 - 2a^2u^3 + O(u^4)$. Reduced dynamics: $\dot{u} = h(u) = au^2 - 2a^2u^3 + \dots$. The leading term is au^2 , which is sign-definite for $u \neq 0$. For any $a \neq 0$, the origin is unstable (it is a saddle-node).

Unstable for all $a \neq 0$.

(8) $\dot{x}_1 = -2x_1 - 3x_2 + x_3 + x_3^2, \quad \dot{x}_2 = x_1 + x_1^2 + x_2, \quad \dot{x}_3 = x_1^2$

Linearization: eigenvalues 0 and two with negative real part ($-1/2 \pm i\sqrt{3}/2$). Center variable: $z = x_3$ (with tangent line $x_1 = -z, x_2 = z$). Center manifold: $x_1 = h_1(z) = -z - 2z^2 + O(z^3)$, $x_2 = h_2(z) = z + 2z^2 + O(z^3)$. Reduced dynamics: $\dot{z} = h_1(z)^2 = z^2 + 4z^3 + \dots$. The leading term is $z^2 > 0$ for $z \neq 0$, so the origin is unstable.

Unstable.

8.7

Consider the system (with the typo x_3^3 corrected to x_1^3)

$$\dot{x}_1 = x_1x_2 + ax_1^3 + bx_1x_2^2, \quad \dot{x}_2 = -x_2 + cx_1^2 + dx_1^2x_2.$$

The origin is an equilibrium. The linearisation about the origin gives

$$J = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},$$

with eigenvalues 0 (center direction $u = x_1$) and -1 (stable direction $v = x_2$).

We seek the center manifold $v = h(u)$ and reduce the dynamics to the center manifold.

Let

$$h(u) = \alpha u^2 + \beta u^3 + \gamma u^4 + \dots$$

Substituting into the second equation and using the first to eliminate time derivatives, we obtain: - $\alpha = c$, - $\beta = 0$, - and when $a + c = 0$, we also get $\gamma = cd$ (see the detailed derivation below).

The reduced dynamics on the center manifold is

$$\dot{u} = uh(u) + au^3 + buh(u)^2.$$

General case $a + c \neq 0$ With $h(u) = cu^2 + O(u^4)$, we get

$$\dot{u} = (a + c)u^3 + O(u^5).$$

Thus the stability is governed by the sign of $a + c$: - If $a + c > 0$, then $\dot{u}$ has the same sign as $u \rightarrow$ origin is unstable. - If $a + c < 0$, then $\dot{u}$ has opposite sign to $u \rightarrow$ origin is asymptotically stable.

Case $a + c = 0$ Then $a = -c$. We need the next term. The center manifold expansion gives

$$h(u) = cu^2 + cd u^4 + O(u^5).$$

Substituting into the reduced dynamics yields

$$\dot{u} = (cd + bc^2)u^5 + O(u^7).$$

Therefore: - If $cd + bc^2 < 0$, then $\dot{u}$ has opposite sign to $u \rightarrow$ origin is asymptotically stable. - If $cd + bc^2 > 0$, then $\dot{u}$ has the same sign as $u \rightarrow$ origin is unstable. - If $cd + bc^2 = 0$, the quintic term vanishes; higher-order terms would be needed (degenerate case, not covered by this analysis).

Summary of the five cases

1. $a + c > 0$	$\Rightarrow$ unstable,
2. $a + c < 0$	$\Rightarrow$ asymptotically stable,
3. $a + c = 0, cd + bc^2 < 0$	$\Rightarrow$ asymptotically stable,
4. $a + c = 0, cd + bc^2 > 0$	$\Rightarrow$ unstable,
5. $a + c = 0, cd + bc^2 = 0$	$\Rightarrow$ degenerate; higher-order terms required.

2

Consider the pendulum dynamics

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{mgl}{J_p} \sin(x_1) - \frac{ml}{J_p} \cos(x_1)u.$$

(a)

The kinetic energy is $T = \frac{1}{2}J_p x_2^2$. Choosing the potential energy zero at the upright position $x_1 = 0$, we have $U = mgl(1 - \cos x_1)$. Hence the total energy is

$$E(x) = \frac{1}{2}J_p x_2^2 + mgl(1 - \cos x_1).$$

At the upright equilibrium $(x_1, x_2) = (0, 0)$:

$$E_0 = \frac{1}{2}J_p(0)^2 + mgl(1 - \cos 0) = 0.$$

$$\boxed{E_0 = 0}$$

(b)

The proposed control is

$$u = k(E - E_0) \operatorname{sign}(x_2 \cos x_1) = kE \operatorname{sign}(x_2 \cos x_1), \quad k > 0.$$

We use the Lyapunov candidate

$$V(x) = (E(x) - E_0)^2 = E^2.$$

First compute $\dot{E}$:

$$\dot{E} = J_p x_2 \dot{x}_2 + mgl \sin(x_1) x_2.$$

Substituting the dynamics:

$$\dot{E} = J_p x_2 \left(-\frac{mgl}{J_p} \sin x_1 - \frac{ml}{J_p} \cos x_1 u \right) + mgl x_2 \sin x_1 = -ml x_2 \cos x_1 u.$$

Now $\dot{V} = 2E\dot{E}$:

$$\dot{V} = 2E(-ml x_2 \cos x_1 u).$$

Insert the control law $u = kE \operatorname{sign}(x_2 \cos x_1)$:

$$\dot{V} = -2mlk E^2 x_2 \cos x_1 \operatorname{sign}(x_2 \cos x_1) = -2mlk E^2 |x_2 \cos x_1| \leq 0.$$

$$\boxed{\dot{V} = -2mlk E^2 |x_2 \cos x_1| \leq 0}$$

Conclusion: The control strategy forces $E \rightarrow E_0$ for almost all initial conditions, except the exact hanging equilibrium $(x_1, x_2) = (\pm\pi, 0)$ where the control becomes zero and the system remains trapped.

(c)

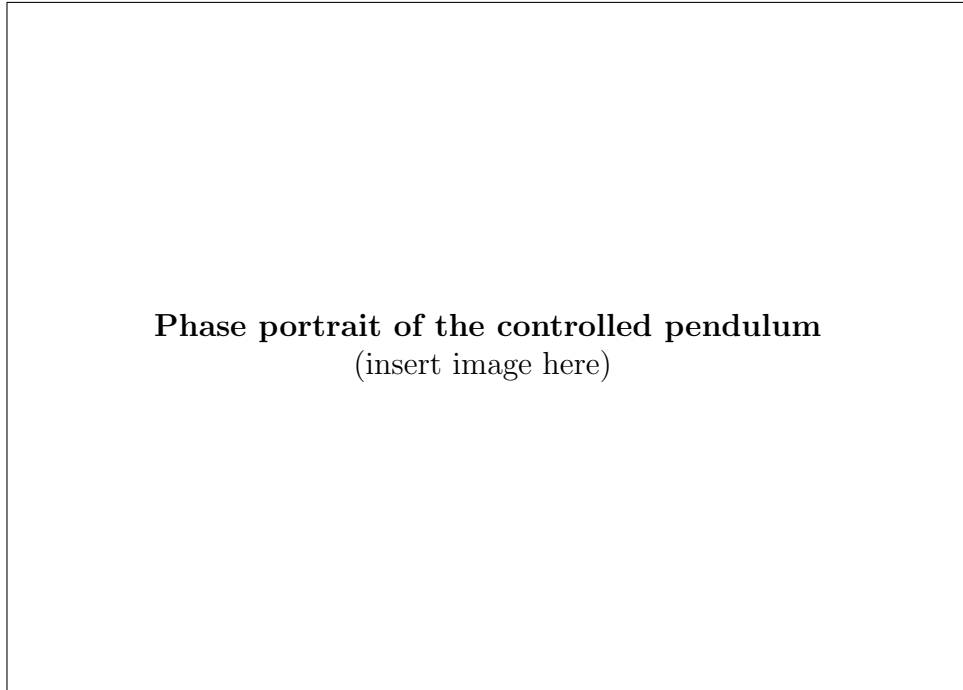


Figure 1: Phase portrait of the controlled inverted pendulum. The green dot is the stable upright position. The red dots are the unstable hanging equilibria.

The closed-loop system has the following properties:

- The upright position $(0, 0)$ is asymptotically stable for almost every initial state.
- The hanging-down equilibria $(\pm\pi, 0)$ are unstable isolated equilibria.
- Trajectories (except those starting exactly at the hanging points) swing up and converge to the upright position as energy is dissipated.

3

Consider

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -ax_1^3 - bx_2 \end{bmatrix}, \quad a > 0, b > 0.$$

(a)

Set the right-hand sides to zero:

$$\dot{x}_1 = 0 \implies x_2 = 0,$$

$$\dot{x}_2 = 0 \implies -ax_1^3 - bx_2 = 0.$$

Substitute $x_2 = 0$:

$$-ax_1^3 = 0 \implies x_1 = 0.$$

Thus the unique equilibrium is

$$\boxed{(x_1, x_2) = (0, 0)}.$$

(b)

Let

$$V(x) = a\frac{x_1^4}{4} + \frac{x_2^2}{2}.$$

Positive definiteness: Since $a > 0$, $V \geq 0$, and $V = 0$ if and only if $x_1 = x_2 = 0$. So V is positive definite.

Derivative along trajectories:

$$\dot{V} = \frac{\partial V}{\partial x_1} \dot{x}_1 + \frac{\partial V}{\partial x_2} \dot{x}_2 = (ax_1^3)(x_2) + (x_2)(-ax_1^3 - bx_2).$$

$$\dot{V} = ax_1^3x_2 - ax_1^3x_2 - bx_2^2 = -bx_2^2 \leq 0.$$

$$\boxed{\dot{V} = -bx_2^2 \leq 0}$$

Hence V is a valid Lyapunov function (positive definite, negative semi-definite derivative), guaranteeing stability of the origin.

(c)

We have $\dot{V} \leq 0$. LaSalle's theorem states that every trajectory converges to the largest invariant set contained in

$$E = \{x \in \mathbb{R}^2 : \dot{V}(x) = 0\}.$$

From $\dot{V} = -bx_2^2$, we get

$$\dot{V} = 0 \iff x_2 = 0.$$

Thus

$$E = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}.$$

Now find the largest invariant subset of E . Suppose a trajectory stays in E . Then $x_2(t) = 0$ for all t . Consequently:

- From $\dot{x}_1 = x_2$, we have $\dot{x}_1 = 0$, so $x_1(t) = c$ (constant).
- From $\dot{x}_2 = -ax_1^3 - bx_2$, with $x_2 = 0$, we get $\dot{x}_2 = -ac^3$.
- For the trajectory to remain in E , we need $\dot{x}_2 = 0$, hence $-ac^3 = 0 \implies c = 0$.

Therefore the only invariant set inside E is the origin itself:

$$\{(0, 0)\}.$$

By LaSalle's principle, every trajectory starting in $\mathbb{R}^2$ converges to the origin. Since V is radially unbounded, the convergence is global:

The origin is globally asymptotically stable.