

NONLINEAR CONTROL SYSTEMS

(PART 7)

@control_notes
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Defining the dynamic system and isolating the equilibrium point

Step 1: The System

$$\dot{x} = f(x)$$

Locally Lipschitz over
a domain $D \subset \mathbb{R}^n$

Step 2: The Target

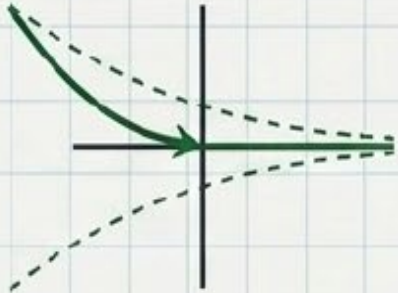
We are diagnosing behavior at an equilibrium point $\bar{x}$, where the system rests: $f(\bar{x}) = 0$

$$y = x - \bar{x}$$

$$\dot{y} = g(y), \text{ where } g(0) = 0$$

By shifting the origin, we universally
analyze stability at 0.

The Stability Spectrum represents increasing levels of gravitational pull

Tier	Concept	Behavioral Rule	Visual Metaphor
1	Stability (Lyapunov)	A particle starting near the origin stays near the origin. It never escapes.	
2	Asymptotic Stability	A particle stays near the origin AND eventually falls into it.	
3	Exponential Stability	A particle is forced into the origin at a guaranteed minimum speed.	

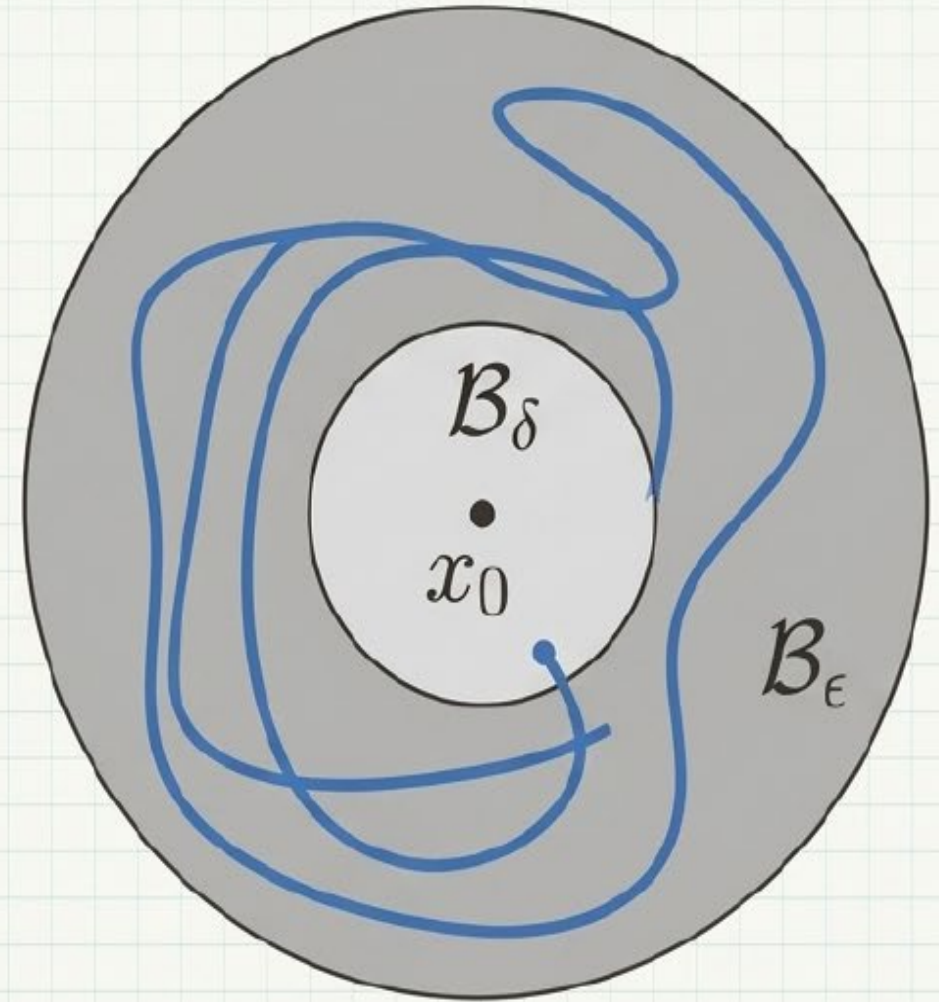
Base Stability ensures the system never breaches a defined boundary

The equilibrium point $x=0$ is stable if for each $\epsilon > 0$, there exists a $\delta > 0$ such that:

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \forall t \geq 0$$

If the system starts inside this inner circle...

...it will never cross this outer boundary.

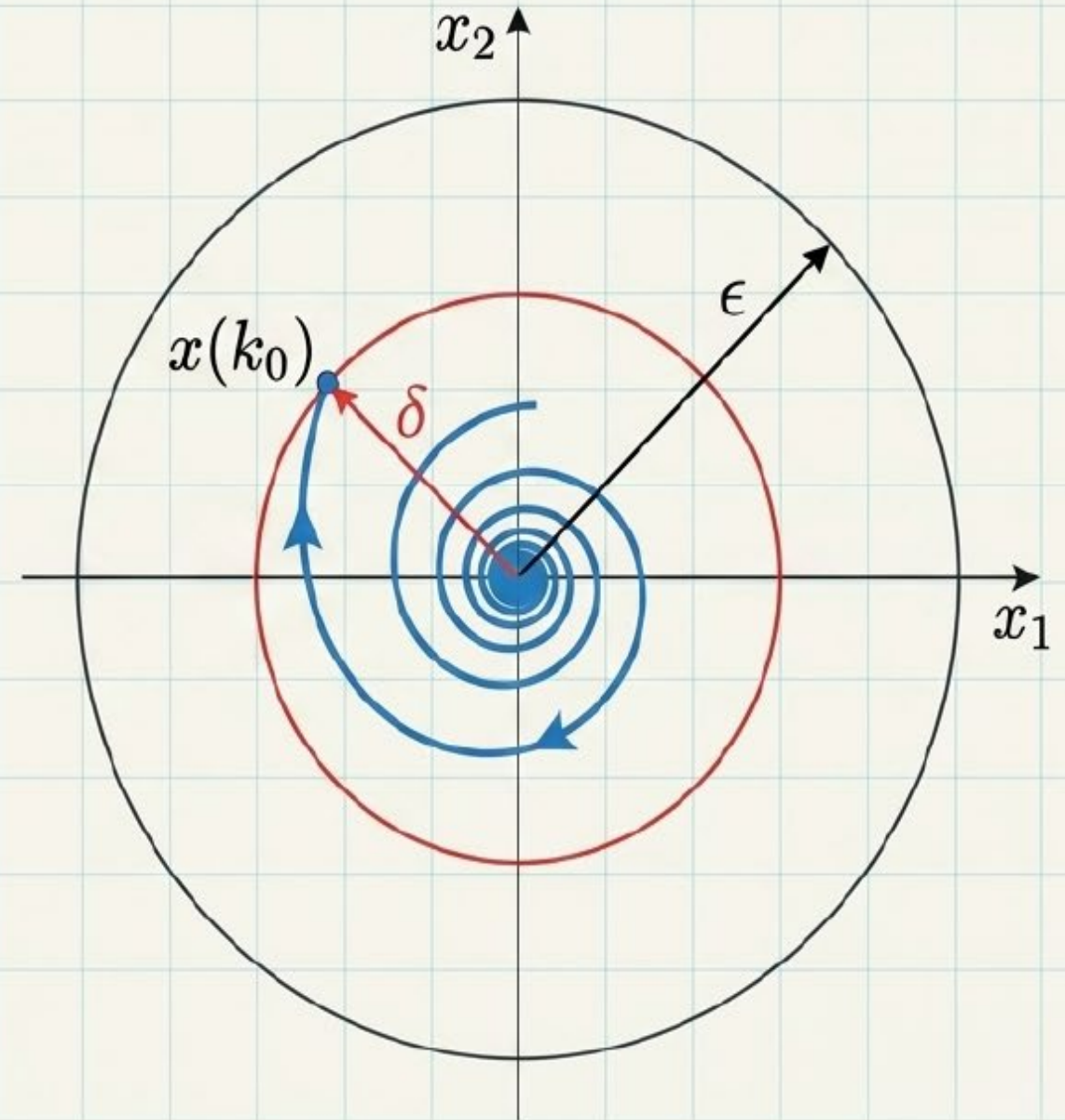


Asymptotic Stability requires containment and eventual convergence

The origin is asymptotically stable if it is stable AND δ can be chosen such that:

$$\|x(0)\| < \delta \Rightarrow \lim_{t \rightarrow \infty} x(t) = 0$$

Mere containment is not enough. The system's energy must dissipate, drawing it inevitably to the zero state.



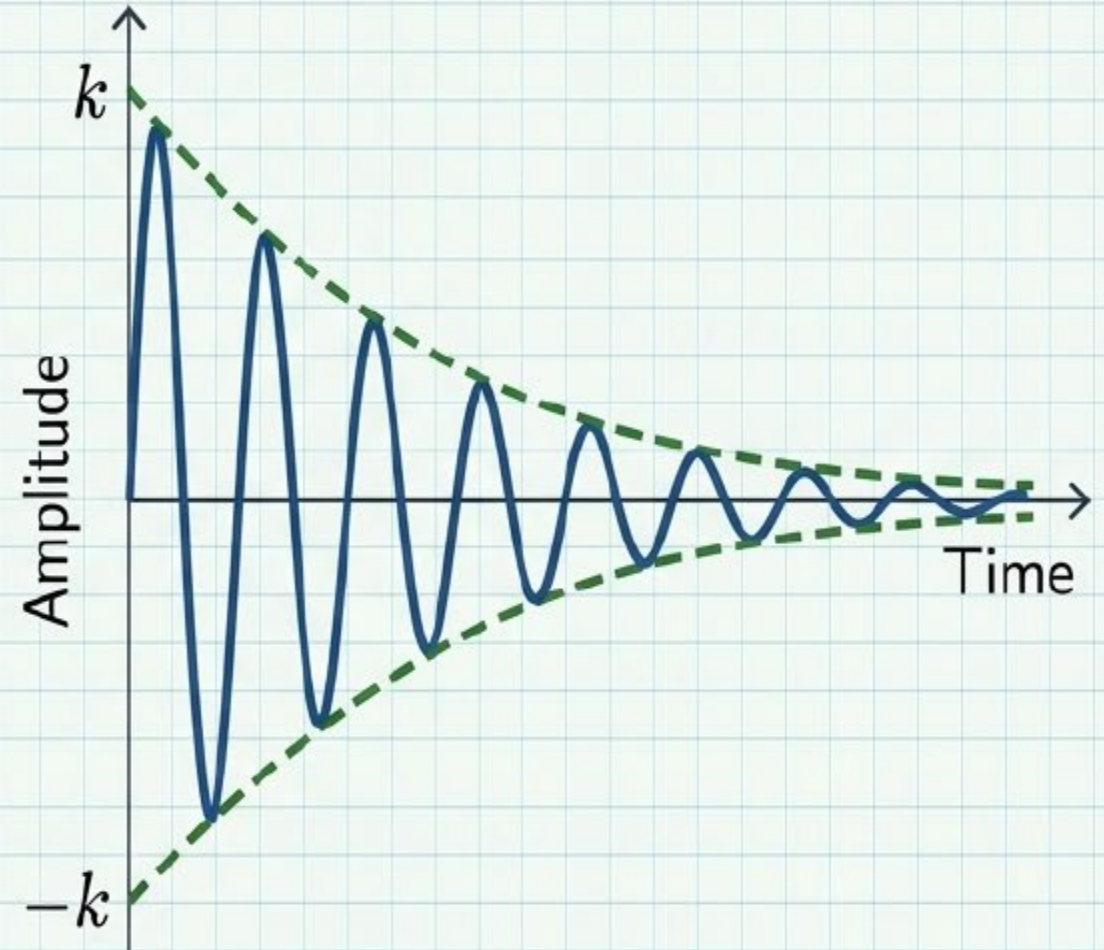
Exponential Stability guarantees a minimum rate of convergence

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \forall t \geq 0$$

$k \geq 1, \lambda > 0$, for all $\|x(0)\| < c$

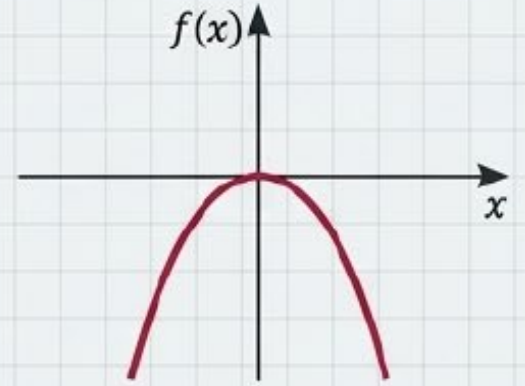
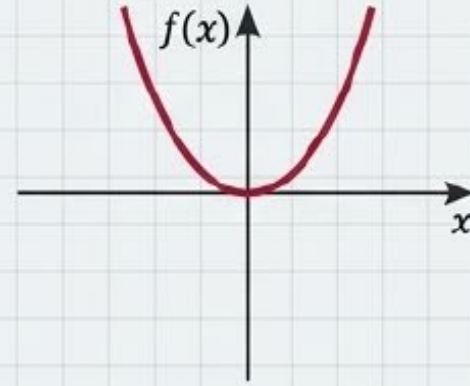
Key Insight

Exponential Stability $\Rightarrow$ Asymptotic Stability. It is a strictly stronger condition because it guarantees the system won't just eventually settle, it will do so bounded by an exponential decay curve.

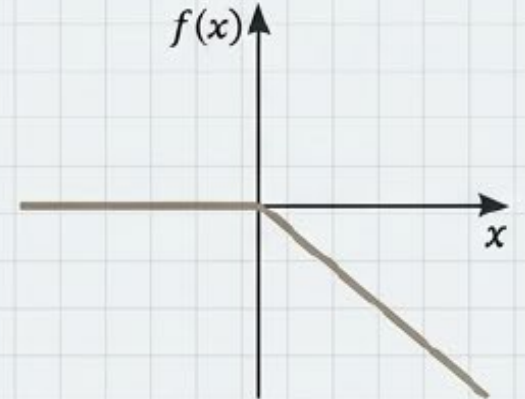
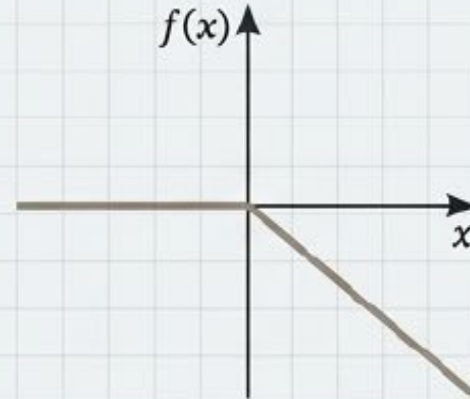
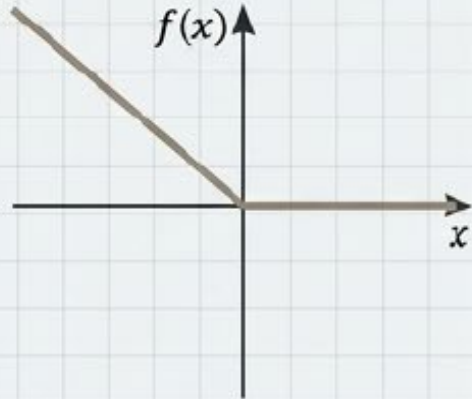


Visualizing the First-Order rules through system geometries

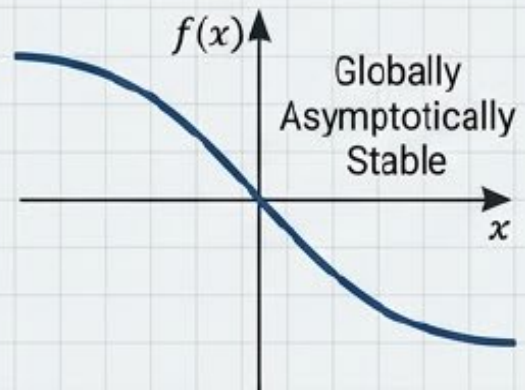
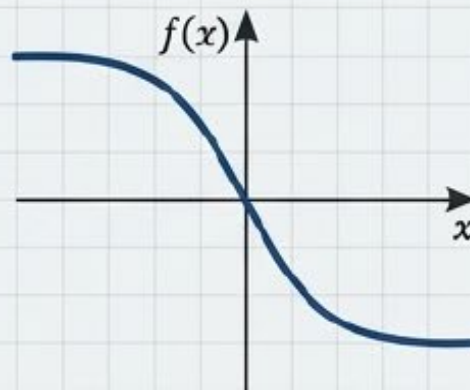
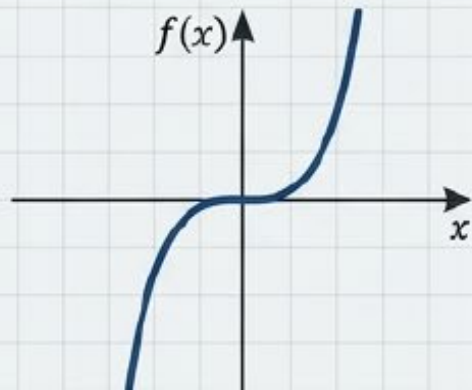
Unstable: $xf(x) > 0$



Stable: $xf(x) \leq 0$

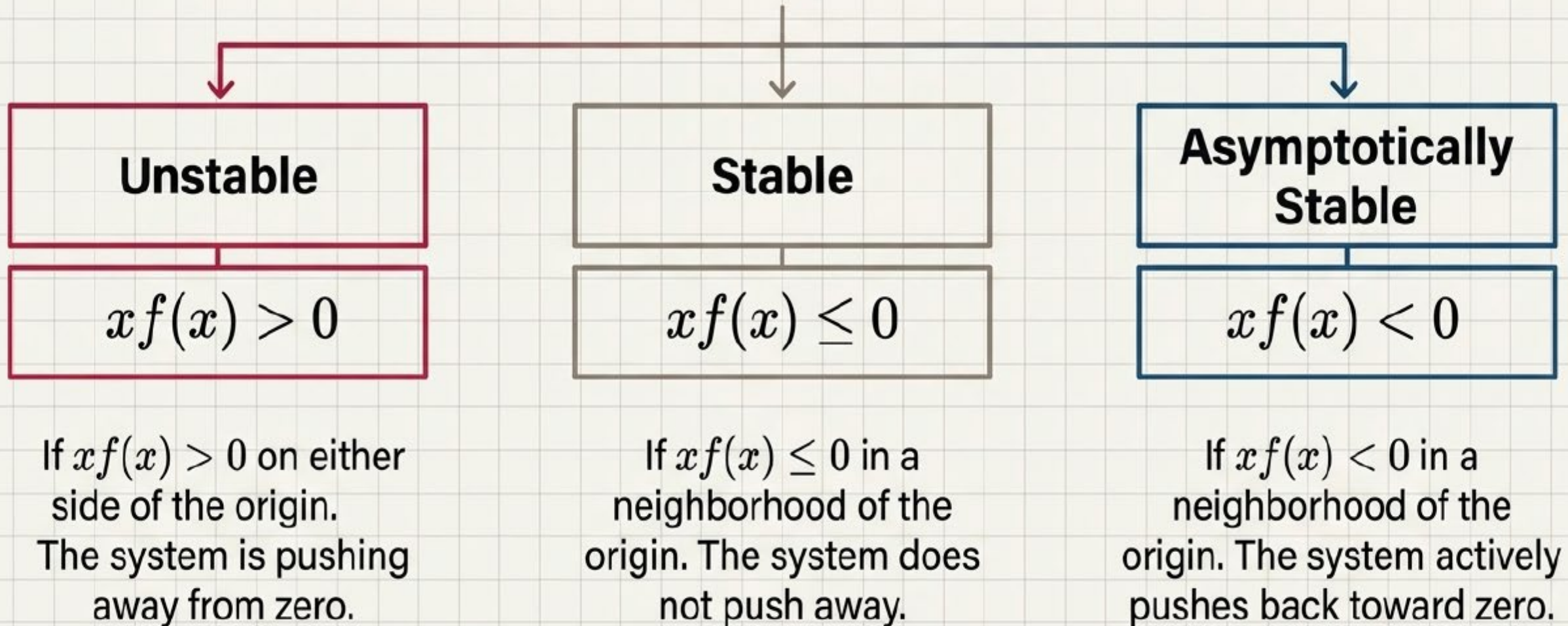


Asymptotically Stable:
 $xf(x) < 0$



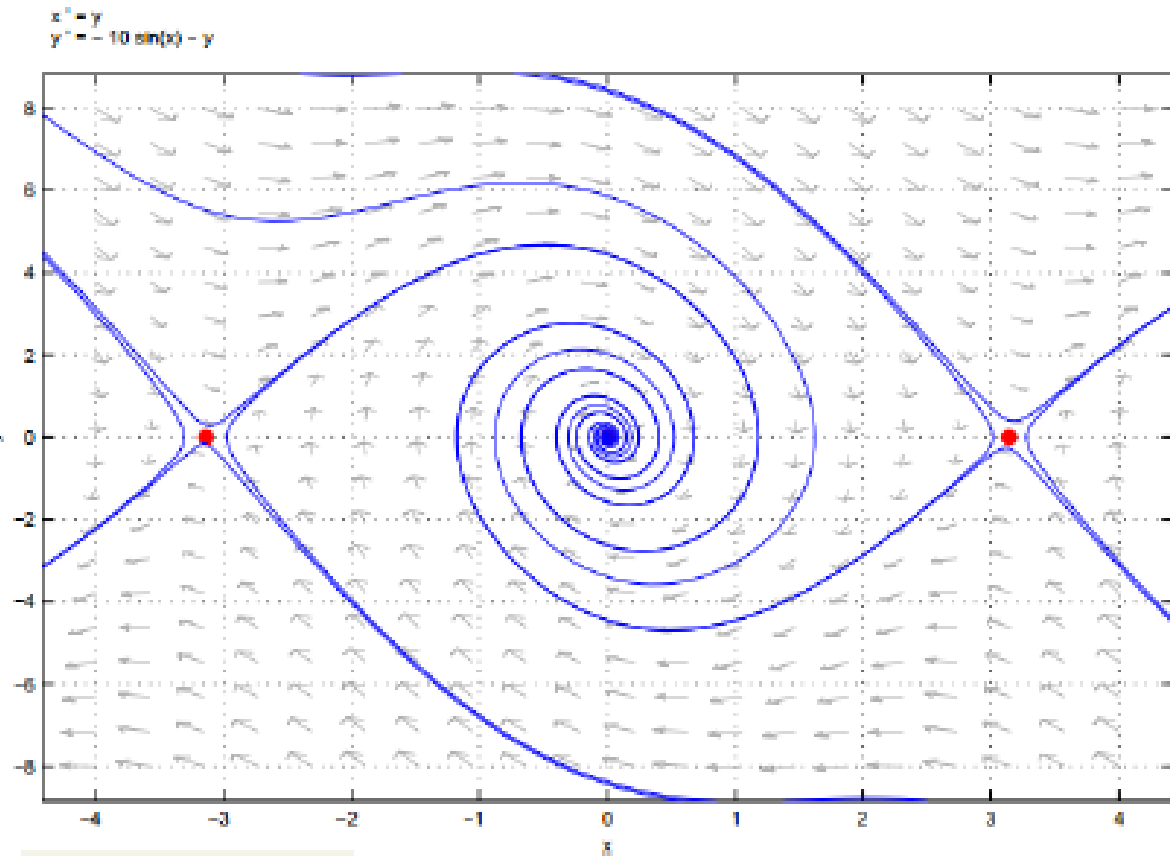
The First-Order Diagnostic: Analyzing system behavior via the sign of $xf(x)$

For 1D systems ($n=1$), stability at the origin is determined entirely by the mathematical sign of $f(x)$ relative to its position x .



Example: Pendulum With Friction

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -10 \sin(x_1) - x_2\end{aligned}$$



Transitioning to complexity: Linear Time-Invariant (LTI) Systems

The Core Formula

$$\dot{x} = Ax$$

Moving beyond 1D, complex systems are modeled using matrices. The behavior is dictated entirely by matrix **A**.

The Universal Solution

$$x(t) = \exp(At)x(0)$$

The Diagnostic Key

Because evaluating $\exp(At)$ directly is computationally complex, stability is diagnosed by unpacking the internal structure of Matrix **A**.

The behavior of the system is permanently encoded in the Eigenvalues (λ) of A.

$$\dot{x} = Ax$$

$$x(t) = \exp(At)x(0)$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & \dots & 0 \\ 0 & \lambda_i & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & 0 \\ \vdots & & & & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 & \lambda_i \end{bmatrix}_{m \times m}$$

$$P^{-1}AP = J = \text{block diag}[J_1, J_2, \dots, J_r]$$

$$\exp(At) = P \exp(Jt) P^{-1} = \sum_{i=1}^r \sum_{k=1}^{m_i} t^{k-1} \exp(\lambda_i t) R_{ik}$$

$$\text{Re}[\lambda_i] < 0 \quad \forall i \Leftrightarrow \text{Asymptotically Stable}$$

$$\text{Re}[\lambda_i] > 0 \text{ for some } i \Rightarrow \text{Unstable}$$

$$\text{Re}[\lambda_i] \leq 0 \quad \forall i \text{ \& } m_i > 1 \text{ for } \text{Re}[\lambda_i] = 0 \Rightarrow \text{Unstable}$$

$$\text{Re}[\lambda_i] \leq 0 \quad \forall i \text{ \& } m_i = 1 \text{ for } \text{Re}[\lambda_i] = 0 \Rightarrow \text{Stable}$$

Theorem: The equilibrium point $x = 0$ of $\dot{x} = Ax$ is stable if and only if all eigenvalues of A satisfy $\text{Re}[\lambda_i] < 0$ and for every eigenvalue with $\text{Re}[\lambda_i] = 0$ and algebraic multiplicity $q_i \geq 2$, $\text{rank}(A - \lambda_i I) = n - q_i$, where n is the dimension of x . The equilibrium point $x = 0$ is globally asymptotically stable if and only if all eigenvalues of A satisfy $\text{Re}[\lambda_i] < 0$

When all eigenvalues of A satisfy $\text{Re}[\lambda_i] < 0$, A is called a *Hurwitz matrix*

Lyapunov's Indirect Method: Using the linear to solve the non-linear

Context

Real physical systems are non-linear: $\dot{x} = f(x)$, with $f(0) = 0$

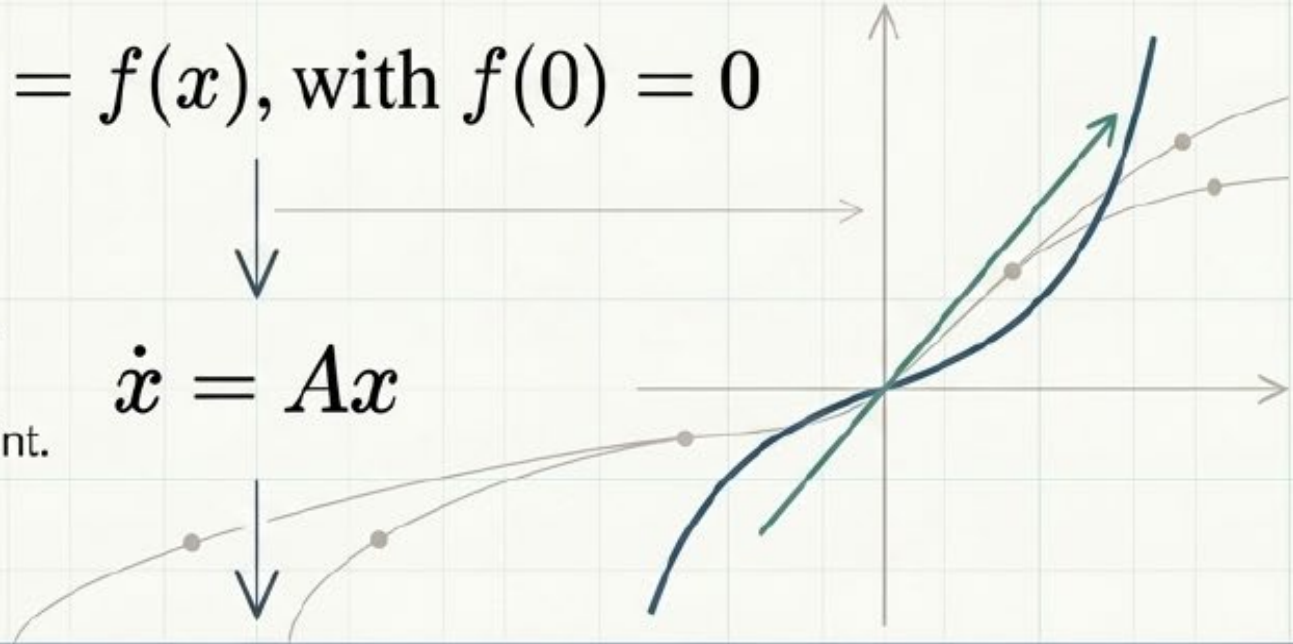
The Mechanism

Instead of solving the complex non-linear equations directly, we extract a linear approximation around the equilibrium point.

The Mathematical Bridge

$$A = \left. \frac{\partial f}{\partial x} \right|_{x=0}$$

We calculate the **Jacobian matrix (A)** to capture the system's local slope at the origin.
We can then instantly test this matrix using our LTI **Eigenvalue Map**.



The Linearization Decision Tree

Calculate Jacobian A &
Extract Eigenvalues (λ)

$\text{Re}[\lambda_i] < 0$ for ALL
eigenvalues.

**The origin is
Exponentially Stable.**

$\text{Re}[\lambda_i] \leq 0$ for all, but
 $\text{Re}[\lambda_i] = 0$ for some.

Linearization Fails.

The linear approximation is too weak. You must return to base definitions and use Lyapunov's Direct Method.

$\text{Re}[\lambda_i] > 0$ for ANY
eigenvalue.

The origin is Unstable.

$$\dot{x} = y$$

$$\dot{y} = \mu(1 - x^2)y - x$$

$$\dot{x} = y$$

$$\dot{y} = -x + \mu y$$



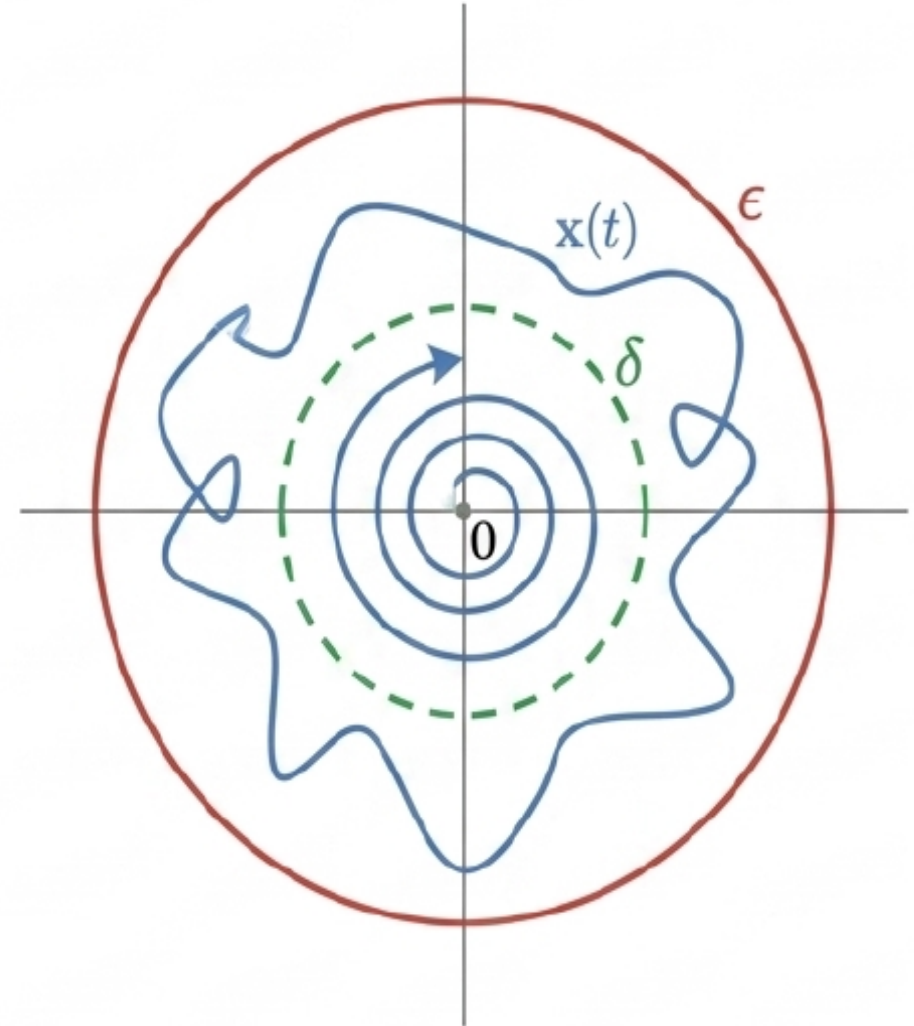
$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\lambda = \frac{1}{2} \left(\mu \pm \sqrt{\mu^2 - 4} \right)$$

Defining Equilibrium and Stability

Analyzing the behavior of system $\dot{x} = f(x, t)$ near equilibrium point $x_e = 0$ starting at time t_0 .

Mathematical Definition	Physical Meaning
Stable $\forall \epsilon > 0, \exists \delta(\epsilon, t_0) > 0$ such that $\ x(t_0)\ < \delta \Rightarrow \ x(t)\ < \epsilon, \forall t \geq t_0$	If you start close enough to the origin (inside δ), you will never leave a defined safety boundary (ϵ).
Convergent $\exists \delta_1(t_0) > 0$ such that $\ x(t_0)\ < \delta_1 \Rightarrow \lim_{t \rightarrow \infty} x(t) = 0$	If you start within a specific distance (δ_1), the trajectory eventually rests at the exact origin.
Asymptotically Stable The equilibrium is both stable and convergent.	



Uniform Stability: Removing Time Dependence

In time-varying systems, does stability depend on when (t_0) the system starts? “Uniform” properties guarantee it does not.

- Uniformly Stable

- The boundary δ depends only on allowed error ε , not on t_0 .

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \text{ s.t. } \|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \forall t \geq t_0 > 0$$

- Uniformly Convergent

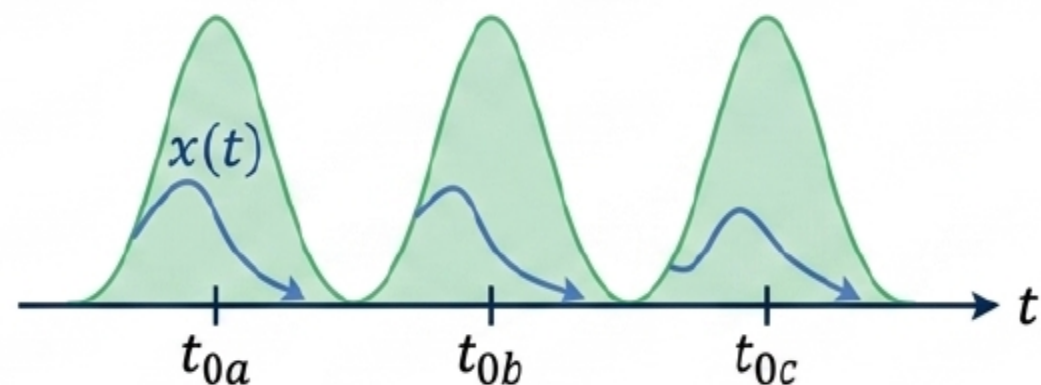
- The convergence threshold δ_1 is a constant, regardless of the start time.

- Uniform Asymptotic Stability

- Both uniformly stable and uniformly convergent.

- Global Uniform Asymptotic Stability

- Every motion in the state space converges to the origin ($\delta_1 \rightarrow \infty$).



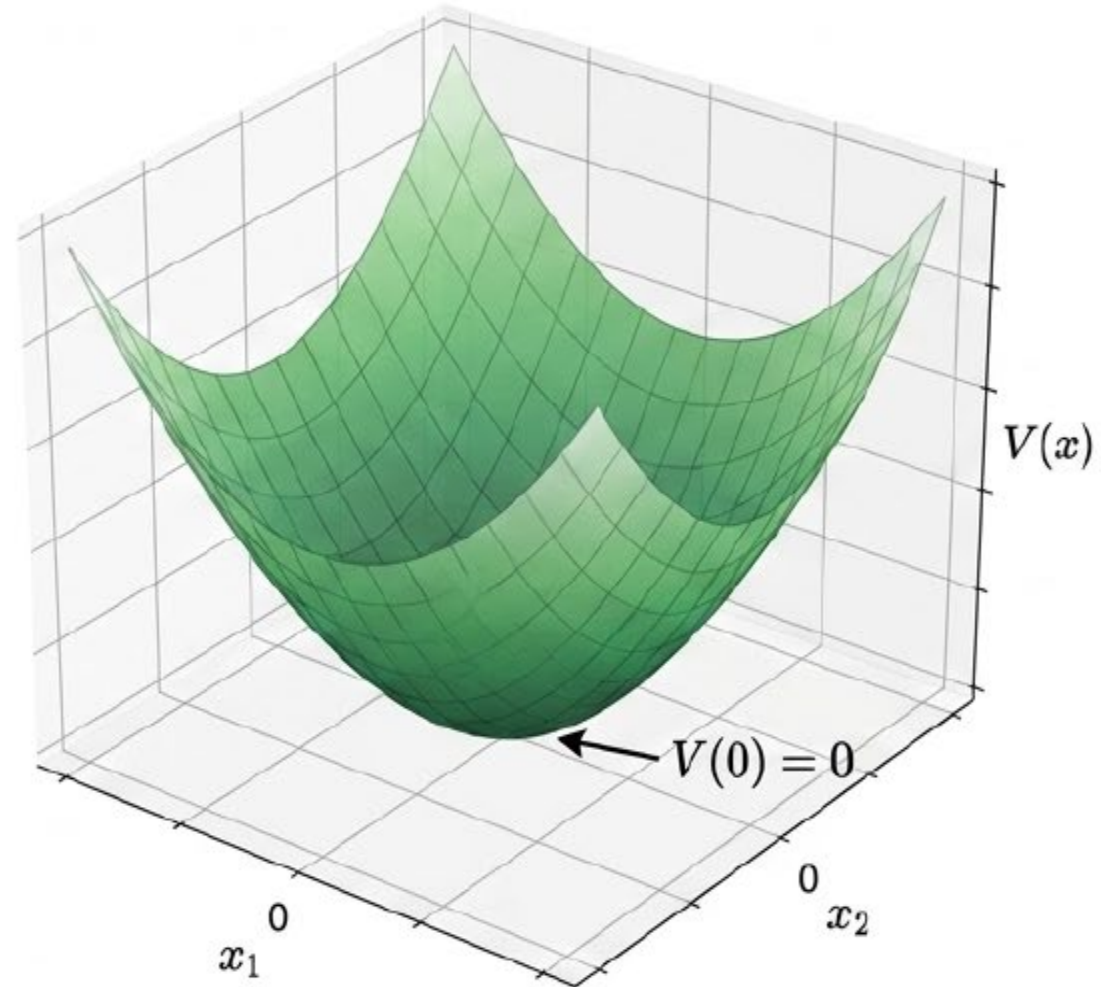
Foundational Functions: Locally Positive Definite (LPD)

Instead of solving differential equations, we define scalar "energy-like" functions, $V(\mathbf{x})$, to analyze system behavior.

A function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ is **LPD** if there exists a radius $R > 0$ such that:

1. $V(0) = 0$ (Zero energy exactly at equilibrium)
2. $V(\mathbf{z}) > 0$ for all $\|\mathbf{z}\| \leq R, \mathbf{z} \neq 0$
(Strictly positive energy everywhere else locally)

$$V_1(x) = x_1^2 + 2x_2^2$$



Categorizing Energy Functions: Definite vs. Semidefinite

Time-dependent functions $W(\mathbf{x}, t)$ are classified by how securely they envelope the state space.

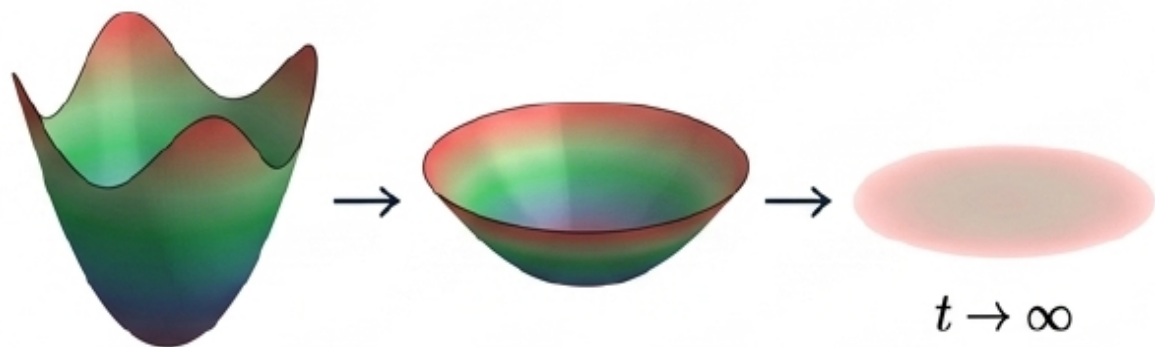
Positive Semidefinite (PSD)

Condition: $W(0, t) = 0$ and $W(\mathbf{x}, t) \geq 0$.

Meaning: Non-negative, but can touch zero outside the origin or decay to zero over time.

Example: $W(\mathbf{x}, t) = (x_1^2 + x_2^2)e^{-\alpha t}$, $\alpha > 0$.

As $t \rightarrow \infty$, the function collapses to 0.



Positive Definite (PD)

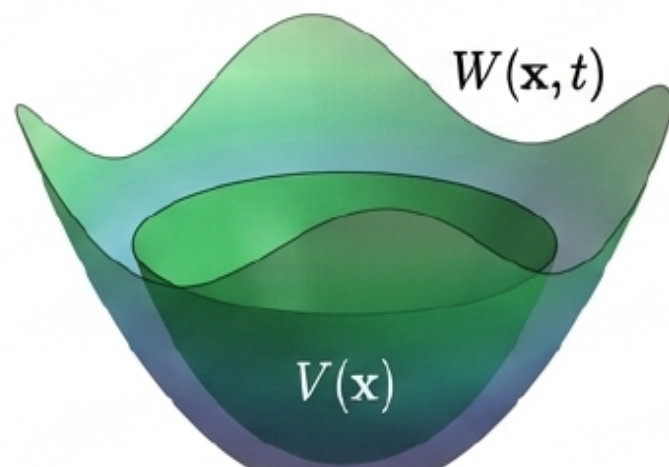
Condition: $W(0, t) = 0$ and $V(\mathbf{x}) \leq W(\mathbf{x}, t)$.

(bounded from below by a time-invariant PD function V).

Meaning: Strictly bounded from below, maintaining a permanent bowl structure over time.

Example: $W(\mathbf{x}, t) = \|\mathbf{x}\|^2 \frac{t^2+1}{t^2+2}$.

With $V_3(\mathbf{x}) = \|\mathbf{x}\|^2$, $\alpha V_3(\mathbf{x}) \leq W(\mathbf{x}, t)$ holds for $\alpha < \frac{1}{2}$.



Advanced Bounds: Decrescent and Radially Unbounded

Decrescent Functions (The Upper Bound)

Concept: The function's decay depends only on state x , bounded from above by a time-invariant PD function.

Math: $|W(x, t)| \leq V_3(x)$ for all $x \in \mathbb{R}^n, t \in \mathbb{R}^+$.

Example: $W(x, t) = \|x\|^2 \frac{t^2+1}{t^2+2}$ is decrescent because $|W(x, t)| \leq \|x\|^2$.

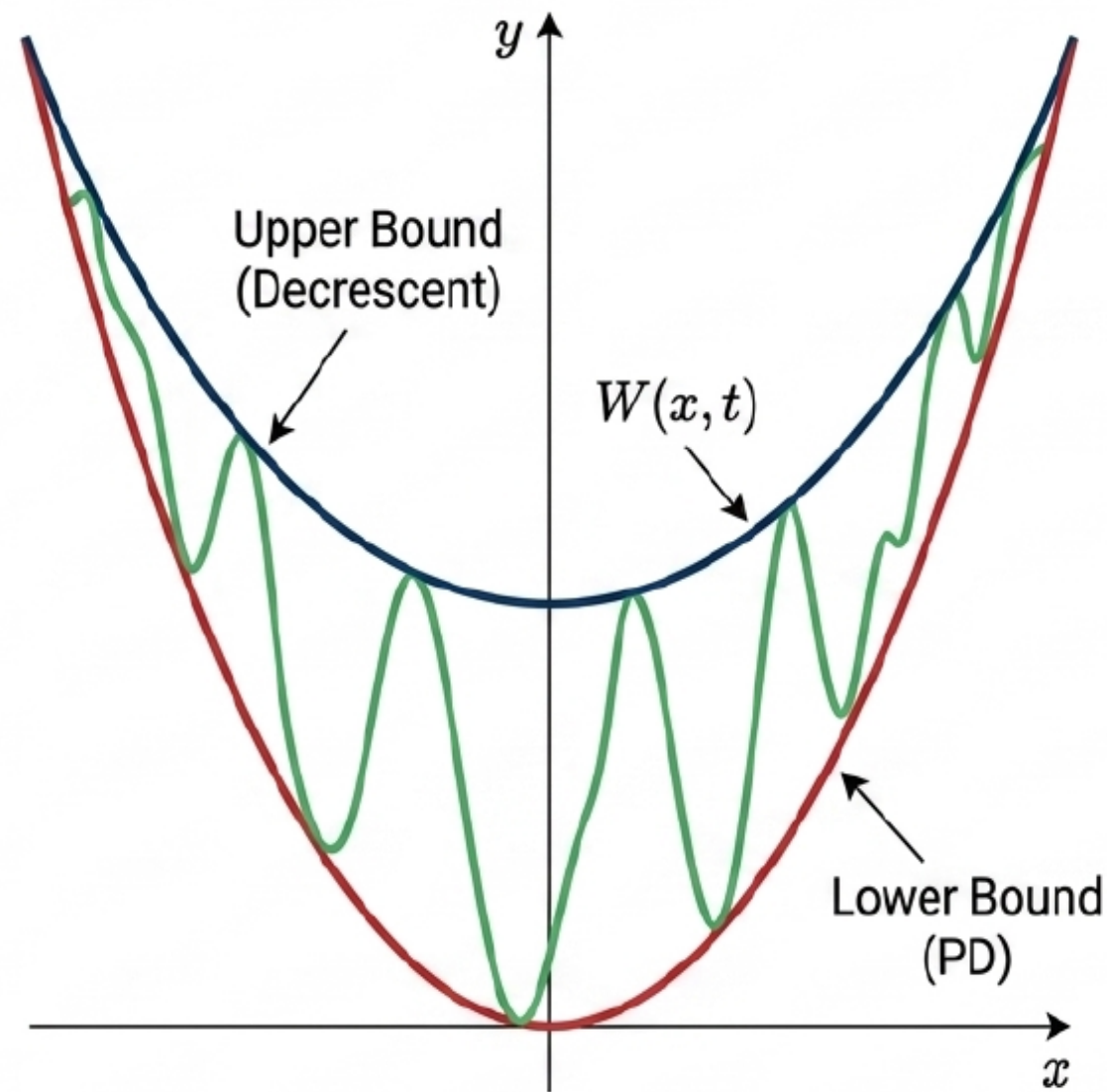
Radially Unbounded Functions (The Infinite Horizon)

Concept: The energy grows infinitely large as distance from the origin reaches infinity. Essential for proving global stability.

Math: $W(x, t) \rightarrow \infty$ as $\|x\| \rightarrow \infty$.

$$W(x, t) = (x_1^2 + x_2^2) / (x_1^2 + 2)$$

$$W(x, t) = (x_1^2 + x_2^2) e^{-\alpha t}, \alpha > 0$$



The Core Mechanic: System Energy Dynamics

Stability is determined by tracking how our chosen energy function $W(\mathbf{x}, t)$ changes along the specific trajectories of our system $\dot{\mathbf{x}} = f(\mathbf{x}, t)$.

1. Total Time Derivative

Apply the chain rule to the energy function.

$$\dot{W}(\mathbf{x}, t) = \frac{\partial W}{\partial t} + \frac{\partial W}{\partial x} \cdot \dot{\mathbf{x}}$$

2. Inject System Dynamics

Substitute the given system equations ($\dot{\mathbf{x}}$) into the derivative.

$$\dot{W}(\mathbf{x}, t) = \frac{\partial W}{\partial t} + \nabla W \cdot f(\mathbf{x}, t)$$

3. Physical Interpretation

If $\dot{W}$ evaluates to a negative value, the “energy” of the system is actively dissipating. The state is sliding down the walls of the topographical basin toward the equilibrium.



Theorem: Stability

The equilibrium $x_e = \mathbf{0}$ at t_0 of $\dot{x} = f(x, t)$ is

- **Stable** if $\exists$ a continuously differentiable **positive definite** $W(x, t)$, $W: D \times \mathcal{R}^+ \rightarrow \mathcal{R}$, $\mathbf{0} \in D$, s. t. $\dot{W}(x, t)$ is negative semidefinite in D
- **Uniformly stable** if $\exists$ a continuously differentiable **positive definite decrescent** $W(x, t)$, $W: D \times \mathcal{R}^+ \rightarrow \mathcal{R}$, $\mathbf{0} \in D$, s. t. $\dot{W}(x, t)$ is negative semidefinite in D

Theorem: Uniform Asymptotic Stability

The equilibrium $x_e = \mathbf{0}$ at t_0 of $\dot{x} = f(x, t)$ is **uniformly asymptotically stable** if $\exists$ a continuously differentiable **positive definite decrescent** $W(x, t)$, $W: D \times \mathcal{R}^+ \rightarrow \mathcal{R}$, $\mathbf{0} \in D$, s. t. $\dot{W}(x, t)$ is negative definite in D

Theorem: Global Uniform Asymptotic Stability

The equilibrium $x_e = \mathbf{0}$ at t_0 of $\dot{x} = f(x, t)$ is **globally uniformly asymptotically stable** if $\exists$ a continuously differentiable **positive definite, decrescent, & radially unbounded** $W(x, t)$, $W: \mathcal{R}^n \times \mathcal{R}^+ \rightarrow \mathcal{R}$, s. t. $\dot{W}(x, t)$ is negative definite $\forall x \in \mathcal{R}^n$

Lyapunov Stability Theorems

The formal links between function properties and equilibrium guarantees.

Function Property	Derivative Bounds	Guarantee
Base Theorem: Stability		
Continuously differentiable, Positive Definite	$\dot{W}(\mathbf{x}, t)$ is negative semidefinite	The equilibrium is Stable
Upgraded Theorem: Uniform Stability		
Continuously differentiable, Positive Definite, AND decrecent	$\dot{W}(\mathbf{x}, t)$ is negative semidefinite	The equilibrium is Uniformly Stable

Guaranteeing Convergence: Asymptotic Theorems

To guarantee absolute convergence, energy dissipation must be strict (definite), and for global convergence, the energy bowl must extend infinitely.

Theorem Progression Matrix: comparison table below as pith ar previous slide

Function Property	Derivative Bounds	Guarantee
Theorem: Uniform Asymptotic Stability		
Positive Definite, Decrescent	$\dot{W}(x, t)$ is negative definite →	Uniform Asymptotic Stability
Theorem: Global Uniform Asymptotic Stability		
Positive Definite, Decrescent, AND Radially Unbounded	 $\dot{W}(x, t)$ is negative definite $\forall x \in \mathbb{R}^n$ →	The system will eventually return to the origin from any state in the state-space.

$$\dot{x}_1 = -x_1 - x_2 e^{-2t}$$

$$\dot{x}_2 = x_1 - x_2$$

Equilibrium: $x_e = \mathbf{0}$ ($f(\mathbf{0}, t) = \mathbf{0}, \forall t \geq t_0, \forall t_0$)

$$W(x, t) = x_1^2 + (1 + e^{-2t})x_2^2$$

$$\begin{aligned} V_1(x) = \|x\|^2 &\leq W(x, t) \leq x_1^2 + 2x_2^2 \\ &= V_2(x) \end{aligned}$$

$$\forall x \in \mathcal{R}^2, \quad \forall t \in \mathcal{R}^+$$

- **Positive definite, decrescent, radially unbounded**

$$\begin{aligned} \dot{W}(x, t) &= \frac{\partial W}{\partial t} + \frac{\partial W}{\partial x} \cdot \dot{x} \\ &= -2x_2^2 e^{-2t} + 2[x_1 \quad x_2(1 + e^{-2t})] \begin{bmatrix} -x_1 - x_2 e^{-2t} \\ x_1 - x_2 \end{bmatrix} \\ &= -2[x_1^2 - x_1 x_2 + x_2^2(1 + 2e^{-2t})] \\ &\leq -2[x_1^2 - x_1 x_2 + x_2^2] = -x^T Q x < 0 \end{aligned}$$

$$Q = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} > 0 \quad x_e = \mathbf{0} \text{ is globally exponentially stable.}$$

Pendulum equation without friction

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -a \sin x_1$$

$$V(x) = a(1 - \cos x_1) + \frac{1}{2}x_2^2$$

$$\dot{V}(x) = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = ax_2 \sin x_1 - ax_2 \sin x_1 = 0$$

The origin is stable

Since $\dot{V}(x) \equiv 0$, the origin is not asymptotically stable

Pendulum equation with friction

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -a \sin x_1 - bx_2$$

$$V(x) = a(1 - \cos x_1) + \frac{1}{2}x_2^2$$

$$\dot{V}(x) = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = -bx_2^2$$

The origin is stable

$\dot{V}(x)$ is not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1

The conditions of Lyapunov's theorem are only sufficient. Failure of a Lyapunov function candidate to satisfy the conditions for stability or asymptotic stability does not mean that the equilibrium point is not stable or asymptotically stable. It only means that such stability property cannot be established by using this Lyapunov function candidate

$$\begin{aligned}
 V(x) &= \frac{1}{2}x^T P x + a(1 - \cos x_1) \\
 &= \frac{1}{2}[x_1 \ x_2] \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + a(1 - \cos x_1)
 \end{aligned}$$

$$p_{11} > 0, \quad p_{11}p_{22} - p_{12}^2 > 0$$

$$\begin{aligned}
 \dot{V}(x) &= (p_{11}x_1 + p_{12}x_2 + a \sin x_1) x_2 \\
 &\quad + (p_{12}x_1 + p_{22}x_2) (-a \sin x_1 - bx_2) \\
 &= a(1 - p_{22})x_2 \sin x_1 - ap_{12}x_1 \sin x_1 \\
 &\quad + (p_{11} - p_{12}b) x_1 x_2 + (p_{12} - p_{22}b) x_2^2
 \end{aligned}$$

$$p_{22} = 1, \quad p_{11} = bp_{12} \Rightarrow 0 < p_{12} < b, \quad \text{Take } p_{12} = b/2$$

$$\dot{V}(x) = -\frac{1}{2}abx_1 \sin x_1 - \frac{1}{2}bx_2^2 \quad D = \{x \in R^2 \mid |x_1| < \pi\}$$

$$V(x) = x^T P x = \sum_{i=1}^n \sum_{j=1}^n p_{ij} x_i x_j, \quad P = P^T$$

$$\lambda_{\min}(P) \|x\|^2 \leq x^T P x \leq \lambda_{\max}(P) \|x\|^2$$

$P \geq 0$ (Positive semidefinite) if and only if $\lambda_i(P) \geq 0 \forall i$

$P > 0$ (Positive definite) if and only if $\lambda_i(P) > 0 \forall i$

$V(x)$ is positive definite if and only if P is positive definite

$V(x)$ is positive semidefinite if and only if P is positive semidefinite

Linear Systems

$$\dot{x} = Ax$$

$$V(x) = x^T P x \quad P = P^T > 0$$

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P)x \stackrel{\text{def}}{=} -x^T Q x$$

If $Q > 0$, then A is Hurwitz

$$PA + A^T P = -Q$$

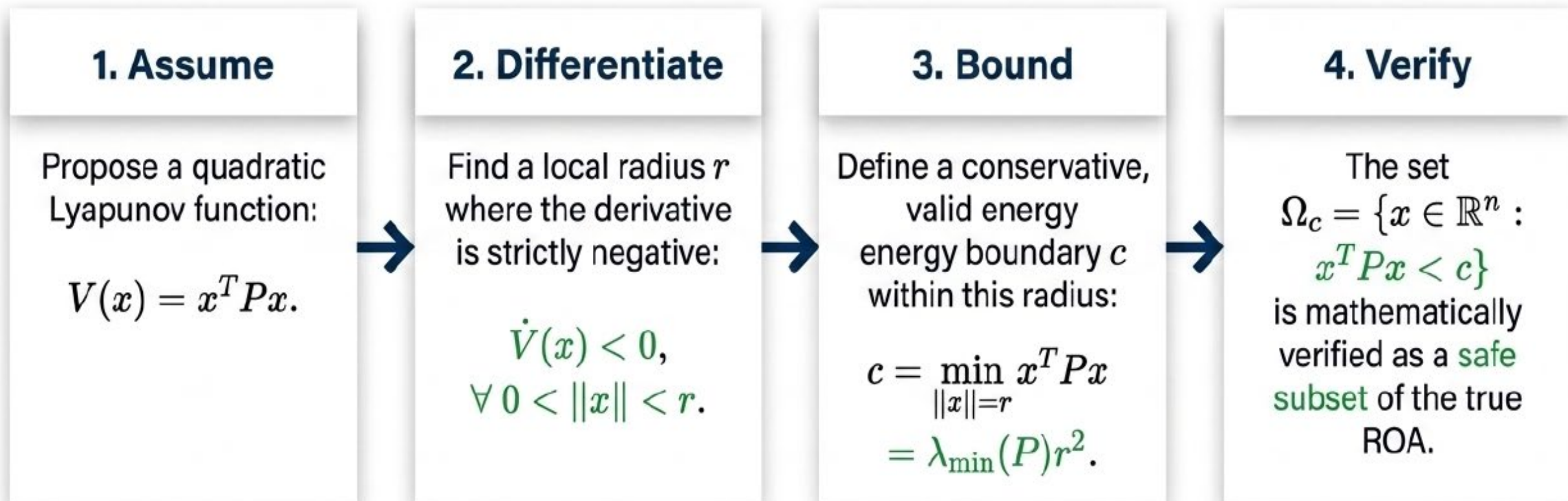
Theorem A matrix A is Hurwitz if and only if for any $Q = Q^T > 0$ there is $P = P^T > 0$ that satisfies the Lyapunov equation

$$PA + A^T P = -Q$$

Moreover, if A is Hurwitz, then P is the unique solution

Practical Application: Estimating the Region of Attraction

For non-global systems, we must calculate the Region of Attraction (ROA)—the localized safe zone where initial conditions mathematically guarantee convergence.



Case Study Step 1: System Definition & Linearization

The Nonlinear System

$$\dot{x}_1 = -x_2$$

$$\dot{x}_2 = x_1 + (x_1^2 - 1)x_2$$

Linearize at
Origin

Linearization Analysis

Compute the Jacobian matrix evaluated at origin:

$$A = \left[\frac{\partial f}{\partial x} \right]_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

Eigenvalue Check: The eigenvalues are $(-1 \pm j\sqrt{3})/2$.

Conclusion: Because the real parts are negative, the origin is locally asymptotically stable. We now proceed to find the size of this stable region.

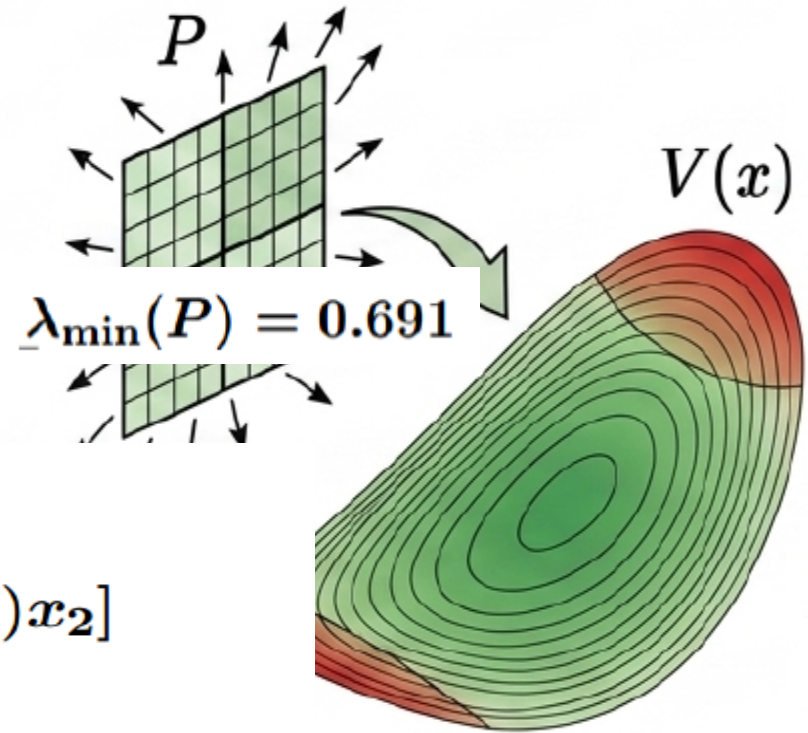
Case Study Step 2: Formulating the Lyapunov Function

We solve the algebraic Lyapunov equation to generate our specific energy matrix P .

The Equation: $PA + A^T P = -Q$

Assumption: Choose identity matrix $Q = I$.

Solving: $PA + A^T P = -I$ yields matrix $P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$



$$V(x) = x^T P x = 1.5x_1^2 - x_1x_2 + x_2^2$$

$$\begin{aligned} \dot{V}(x) &= (3x_1 - x_2)(-x_2) + (-x_1 + 2x_2)[x_1 + (x_1^2 - 1)x_2] \\ &= -(x_1^2 + x_2^2) - (x_1^3x_2 - 2x_1^2x_2^2) \end{aligned}$$

$$|x_1| \leq \|x\|, \quad |x_1x_2| \leq \frac{1}{2}\|x\|^2, \quad |x_1 - 2x_2| \leq \sqrt{5}\|x\|$$

$$\Rightarrow \dot{V}(x) \leq -\|x\|^2 + |x_1| |x_1x_2| |x_1 - 2x_2| \leq -\|x\|^2 + \frac{\sqrt{5}}{2}\|x\|^4$$

Case Study Step 4: Calculating the Boundary Constant

1. Find the Safe Radius (r):

Stability requires $\dot{V}(x) < 0$. Set the bound to zero:

$$-\|x\|^2 + \frac{\sqrt{5}}{2}\|x\|^4 < 0.$$

Valid condition: $0 < \|x\|^2 < \frac{2}{\sqrt{5}}$.

Therefore, defined maximum radius squared: $r^2 = \frac{2}{\sqrt{5}}$

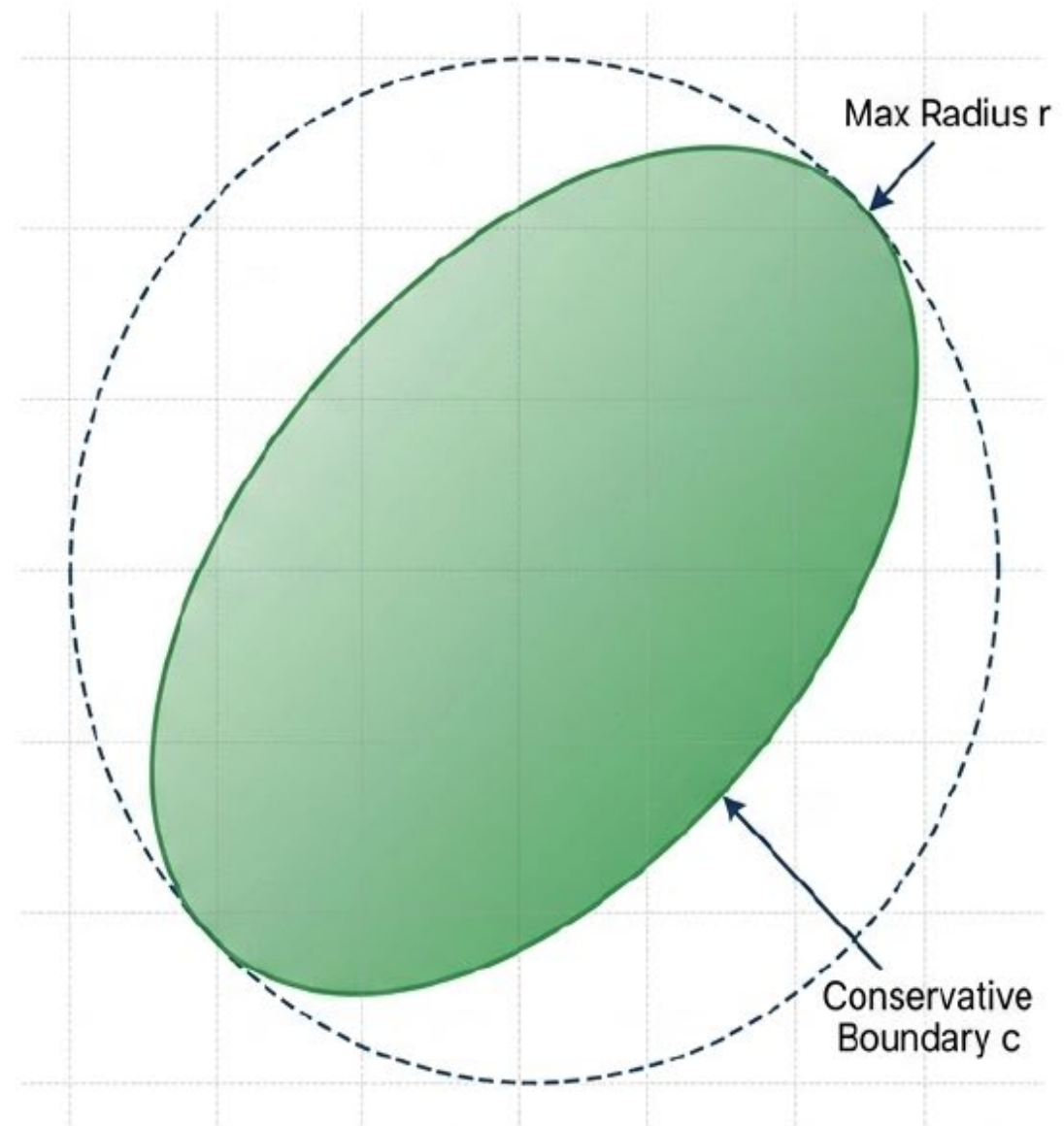
2. Calculate Constant (c):

Use established formula:

$$c = \lambda_{\min}(P)r^2.$$

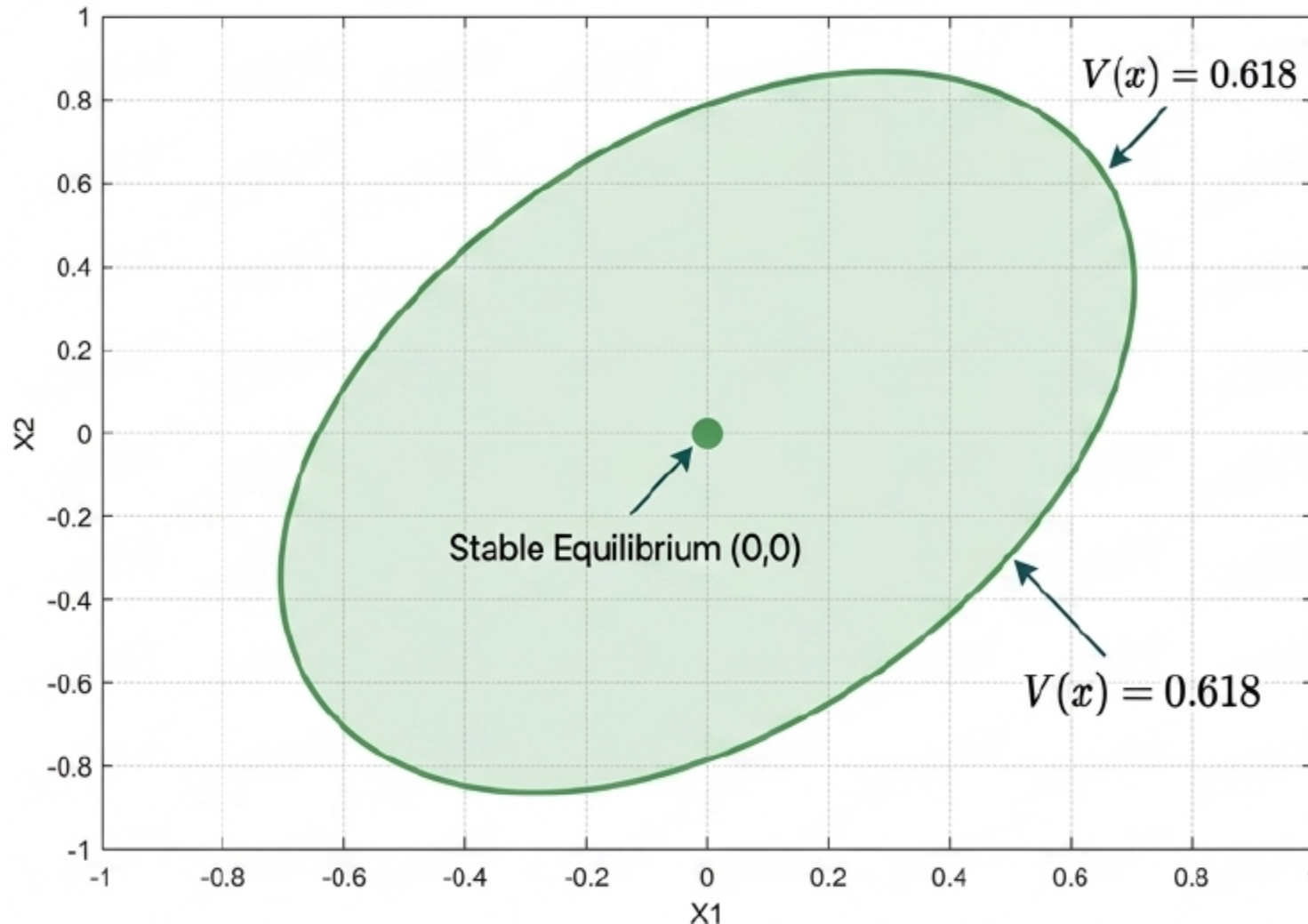
$$c = 0.691 \times \frac{2}{\sqrt{5}} = 0.618$$

Conclusion: The specific set $\{V(x) < 0.618\}$ provides an **absolute mathematical guarantee of convergence**.



Case Study Conclusion: The Region of Attraction

The inequality $1.5x_1^2 - x_1x_2 + x_2^2 < 0.618$ defines a specific geometric safe-zone in the state space.



Key Takeaways

- **Inside the Boundary:** Any system trajectory starting inside this green ellipse is mathematically guaranteed to spiral inward to the origin.
- **Conservative Nature:** The system might still be stable outside this boundary, but the chosen Lyapunov function only guarantees safety strictly inside this computed geometry.
- **Engineering Value:** We verified a functional safety envelope without ever integrating the nonlinear differential equations.